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Vertex-disjoint cycles of the same length in tournaments. (English) Zbl 1533.05137

J. Graph Theory 102, No. 4, 666-683 (2023).

For a digraph D, let $\delta^+(D)$ denote its minimum outdegree. A collection of digraphs on a given vertex set are said to be vertex-disjoint (or disjoint, in short) if no two of them share a common vertex.

J. C. Bermond and *C. Thomassen* conjectured [ibid. 5, 1–43 (1981; Zbl 0458.05035)] that for any $k \geq 1$, any digraph D with $\delta^+(D) \geq 2k-1$ has at least k disjoint cycles. For the class of tournaments, *J. Bang-Jensen* et al. [ibid. 75, No. 3, 284–302 (2014; Zbl 1292.05119)] proved that the conjecture is true, and there are at least k disjoint 3-cycles in tournaments T with $\delta^+(T) \geq 2k-1$. Furthermore, they proved that for every $\epsilon > 0$, if k is large enough, then there are at least k disjoint cycles in any tournament T with $\delta^+(T) \geq (\frac{3}{2}+\epsilon)k$, where the constant $\frac{3}{2}$ is best possible.

Similarly, one can ask about the existence of q-cycles in tournaments, for any fixed $q \geq 3$. *N. Lichiardopol* [Discrete Math. 310, No. 19, 2567–2570 (2010; Zbl 1227.05157)] conjectured that for every $q \geq 3$ and $k \geq 1$, any tournament T with $\delta^+(T) \geq (q-1)k-1$ contains k disjoint q-cycles, and this was subsequently proved by the work of *Y. Wang* et al. [Acta Math. Appl. Sin., Engl. Ser. 38, No. 2, 271–281 (2022; Zbl 1486.05121)], *F. Ma* et al. [Electron. J. Comb. 27, No. 2, Research Paper P2.52, 20 p. (2020; Zbl 1441.05093)], and *S. Zhu* [Shandong University (Master's Thesis) (2019)].

In this paper, the authors prove an analogue of the asymptotic result of Bang-Jensen et al. [loc. cit.], extending the theorem from the case $q = 3$ to all $q \geq 4$, which also answers affirmatively a question raised in [Ma et al., loc. cit.]. Precisely, they prove that for each $q \geq 4$, for every $\epsilon > 0$, if k is large enough, then there are at least k disjoint q-cycles in any tournament T with $\delta^+(T) \geq (\frac{q}{2}+\epsilon)k$, where the constant $\frac{q}{2}$ is best possible.

Since these asymptotic results do not consider the case when the minimum degree of the tournament is $qk/2$, the authors also raise the following question: for given integers $k \geq 1$ and $q \geq 3$, does any (regular) tournament T for which $\delta^+(T) \geq qk/2$ have at least k disjoint cycles of length q?

Reviewer: Brahadeesh Sankarnarayanan (Mumbai)

MSC:

05C38 Paths and cycles
05C12 Distance in graphs
05C20 Directed graphs (digraphs), tournaments

Keywords:

disjoint cycles; minimum outdegree; tournament

Full Text: DOI

References:

[1] Alon, N., Disjoint directed cycles, J. Combin. Theory Ser. B, 68, 2, 167-178, 1996 · Zbl 0861.05037

[2] Bang-Jensen, J.; Gutin, G., Digraphs: Theory, algorithms and applications. 2nd ed., 2009, Springer-Verlag: London · Zbl 1170.05002

[3] Bang-Jensen, J.; Bessy, S.; Thomassé, S., Disjoint 3-cycles in tournaments: A proof of the Bermond-Thomassen conjecture for tournaments, J. Graph Theory., 75, 3, 284-302, 2014 · Zbl 1292.05119

[4] Bensmail, J.; Harutyunyan, A.; Le, N.-K.; Li, B.; Lichiardopol, N., Disjoint cycles of different lengths in graphs and digraphs, Electron. J. Combin., 24, 4, 1-24, 2017 · Zbl 1376.05038

[5] Bermond, J.-C.; Thomassen, C., Cycles in digraphs—A survey, J. Graph Theory., 5, 1, 1-43, 1981 · Zbl 0458.05035

[6] Bucić, M., An improved bound for disjoint directed cycles, Discrete Math., 341, 8, 2231-2236, 2018 · Zbl 1388.05076

[7] Cuckler, B.

[8] Henning, M. A.; Yeo, A., Vertex disjoint cycles of different length in digraphs, SIAM J. Discrete Math., 26, 2, 687-694, 2012

· Zbl 1248.05079

[9] Keevash, P.; Sudakov, B., Triangle packings and 1-factors in oriented graphs, J. Combin. Theory Ser. B., 99, 4, 709-727, 2009 · Zbl 1208.05038

[10] Li, L.; Molla, T., Cyclic triangle factors in regular tournaments, Electron. J. Combin., 26, 4, 4-24, 2019 · Zbl 1432.05082

[11] Lichiardopol, N., Vertex-disjoint directed cycles of prescribed length in tournaments with given minimum outdegree and indegree, Discrete Math., 310, 19, 2567-2570, 2010 · Zbl 1227.05157

[12] Lichiardopol, N., Proof of a conjecture of Henning and Yeo on vertex-disjoint directed cycles, SIAM J. Discrete Math., 28, 3, 1618-1627, 2014 · Zbl 1305.05109

[13] Lichiardopol, N.; Pór, A.; Sereni, J.-S., A step toward the Bermond-Thomassen conjecture about disjoint cycles in digraphs, SIAM J. Discrete Math., 23, 2, 979-992, 2009 · Zbl 1191.05054

[14] Ma, F.; Yan, J., An improvement of Lichiardopol's theorem on disjoint cycles in tournaments, Appl. Math. Comput., 347, 162-168, 2019 · Zbl 1428.05124

[15] Ma, F.; West, D.-B.; Yan, J., Lichiardopol's conjecture on disjoint cycles in tournaments, Electron. J. Combin., 27, 2, 1-20, 2020 · Zbl 1441.05093

[16] Thomassen, C., Disjoint cycles in digraphs, Combinatorica., 3, 3-4, 393-396, 1983 · Zbl 0527.05036

[17] Wang, Y.; Yan, J.; Zhu, S., On disjoint cycles and the same length in tournaments, Acta Math. Appl. Sin.—E., 38, 2, 271-281, 2022 · Zbl 1486.05121

[18] Yuster, R., Combinatorial and computational aspects of graph packing and graph decomposition, Comput. Sci. Rev, 1, 1, 12-26, 2007 · Zbl 1302.05149